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A Category-Theoretic Approach to Social Network Analysis

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Abstract

In this paper we introduce a category-theoretic formalisation of social network analysis. This generalises traditional graph-theoretic formalisations and facilitates a formal approach to statements and beliefs about social networks. We describe a formal semantics for belief in social networks, and we illustrate our formalisation by a case study drawn from organisational structure in the Gulf War.

1 Introduction

Social Network Analysis [14] is an approach to analysing organisations focusing on the relationships between people and/or groups as the most important aspect. Going back to the 1950's, it is characterised by adopting mathematical techniques especially from graph theory [7,9]. It has applications in organisational psychology, sociology and anthropology.

The first goal of Social Network Analysis is to visualise communication and other relationships between people and/or groups by means of diagrams. The second goal is to study the factors which influence relationships and to study the correlations between relationships. The third goal is to draw out implications of the relational data, including bottlenecks where multiple information flows funnel through one person or section (slowing down work processes) and situations where information flows does not match formal group structure. The fourth and most important goal of Social Network Analysis is to make recommendations to improve communication and workflow in an organisation, and (in military terms) to speed up the orient-observe-decide-act (OODA) loop or decision cycle.

Social Network Analysis provides an avenue for analysing and comparing formal and informal information flows in an organisation, as well as comparing

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information flows with officially defined work processes. In previous work, we have applied Social Network Analysis to military organisations [4]. In this paper, we use *category theory* [1] to model social networks, since this is capable of unifying approaches based on algebra, graph theory, and logic. We have constructed a Java-based tool called CAVALIER, to carry out Social Network Analysis based on this approach.

2 Networks and Link Sets

A social network $\mathcal{N}$ consists of a collection of *nodes* (people, organisations, or groups) $A, B, C, \ldots$ together with a collection of *link sets* $\mathcal{L}(A, B)$ which generalise the idea of a link from A to B. A link set incorporates different attributes of a link, as well as different concepts of distance between nodes. Each link set may be empty (indicating no link) or have the form:

$$\mathcal{L}(A, B) = \{p_1, p_2, \ldots, [\delta_1 : d_1], \ldots, [\delta_n : d_n]\}$$

Here the p_i are *predicates*, which include *basic predicates* b_j and *attributes* $[a_k = v_k]$ where a_k is an *attribute name* and v_k is the corresponding *attribute value*. If $[a = v]$ is a member of a link set then $[a = w]$ cannot be a member for $w \neq v$, i.e. each attribute name can only be associated with one value. Each δ_i is a *distance operator* and d_i is the corresponding distance between A and B measured using δ_i, where $0 \leq d_i < \infty$. If $[\delta : d]$ is a member of a link set then $[\delta : d']$ cannot be a member for $d' \neq d$, i.e. each distance operator can only be associated with one distance value. For each node A we have $\mathcal{L}(A, A) = \{\}$, i.e. there is no link from a node to itself.

As an example, the basic predicate **supervisor** $\in \mathcal{L}(A, B)$ indicates that B is the supervisor of A. There are also distinguished basic predicates **true** and **false** such that for each non-empty link set $\mathcal{L}(A, B)$, **true** $\in \mathcal{L}(A, B)$ but **false** $\notin \mathcal{L}(A, B)$.

Attributes include **colour** corresponding to the colour with which the link should be drawn in a diagram and also **from** and **to** corresponding to node names: if $\mathcal{L}(A, B)$ is non-empty, then $[\textbf{from} = A] \in \mathcal{L}(A, B)$ and $[\textbf{to} = B] \in \mathcal{L}(A, B)$. Our Java-based CAVALIER tool allows the creation, editing, and visualisation of link sets within a network. As well as manual editing using a GUI interface, systematic update commands can be applied. For example, the command `colour := supervisor ? "red" : "grey"` updates the **colour** attribute in each non-empty link set $\mathcal{L}(A, B)$ so that if it is a supervisor link the colour becomes red and otherwise it becomes grey. The tool also includes a statistics package which can analyse relationships between attributes and distance operators. Figure 1 shows an example social network diagram produced by the CAVALIER tool, based on the the ground force structure during the Gulf War [3,8]. Boxes represent division-level units from participating countries, while circles represent commanders. Units on the right were under American control, and those on the left under Saudi control.

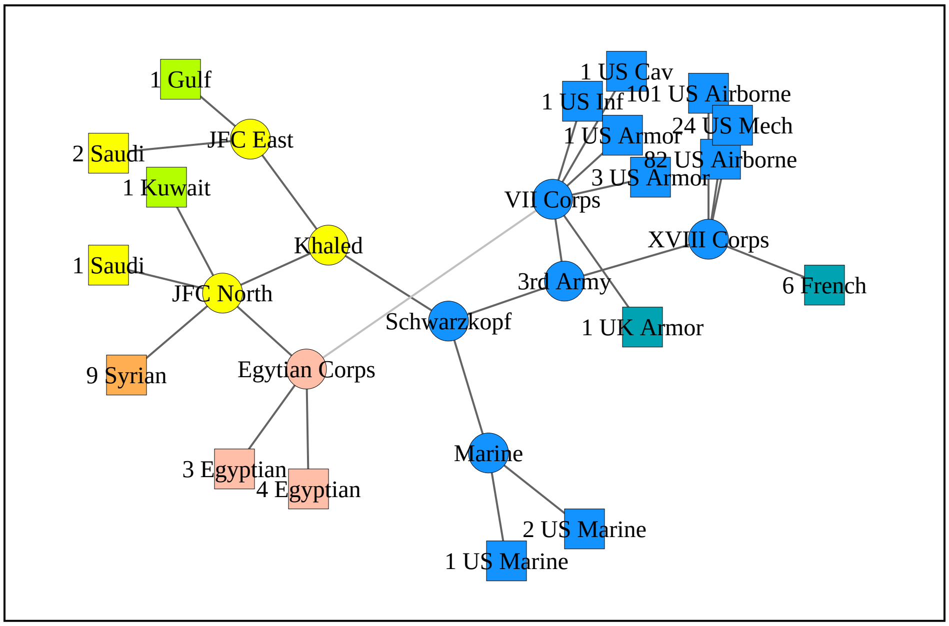

Fig. 1. Social Network for Gulf War Ground Forces

Link sets are closed under the logical operators $\wedge$ (and) and $\vee$ (or) on predicates, i.e. for each link set $\mathcal{L}(A, B)$:

$$p \wedge q \in \mathcal{L}(A, B) \quad \text{if and only if} \quad p \in \mathcal{L}(A, B) \quad \text{and} \quad q \in \mathcal{L}(A, B)$$
$$p \vee q \in \mathcal{L}(A, B) \quad \text{if and only if} \quad p \in \mathcal{L}(A, B) \quad \text{or} \quad q \in \mathcal{L}(A, B)$$

Definition 2.1 If $\mathcal{L}(A, B) = \mathcal{L}(B, A)$ for each A and B, we say that the network is *symmetric* and draw the links using lines rather than arrows.

Definition 2.2 We write $\delta \in S$ if $[\delta : d] \in S$ for some $d < \infty$, and $\delta \notin S$ otherwise. We also define the direct (single step) distance along a link under the distance operator δ:

$$\begin{aligned} S \# \delta &= d \quad \text{if } [\delta : d] \in S \\ &= \infty \quad \text{if } \delta \notin S \end{aligned}$$

A distance of ∞ between two nodes means that (under a particular distance operator), there is no direct connection between the two nodes. The simplest use of distance operators has $[\delta : 1]$ for each link set, i.e. each link is considered to have length 1 under the distance operator δ. Alternatively, link distances can reflect the strength of a link, under various criteria. For example, we may use distance operators based on the amount of formal and informal communication between people. If we have a definition of the strength s of a link, we associate this with a distance operator $[\delta : 1/s]$, where $1/0 = \infty$. For military networks, we also reserve one distance operator to refer to the (absolute) difference in rank between two people.

Definition 2.3 We define the *restriction* of a link set to satisfy the predicate p as follows:

$$S \downarrow p = S \quad \text{if } p \in S$$
$$= \{\} \quad \text{if } p \notin S$$

Definition 2.4 We write $p \Rightarrow_{\mathcal{N}} q$ (p implies q) for a specific network $\mathcal{N}$ if $p \in \mathcal{L}(A, B)$ implies $q \in \mathcal{L}(A, B)$ for each link set $\mathcal{L}(A, B)$ in $\mathcal{N}$. We omit the subscripts where the network $\mathcal{N}$ is clear from context. We write $p \Leftrightarrow_{\mathcal{N}} q$ if $p \Rightarrow_{\mathcal{N}} q$ and $q \Rightarrow_{\mathcal{N}} p$, i.e. if p and q occur in exactly the same link sets.

3 Properties of Link Sets

We now turn our attention to properties of link sets.

Proposition 3.1 *For every network $\mathcal{N}$:*

(i) If p logically implies q, then $p \Rightarrow_{\mathcal{N}} q$.

(ii) **true** $\Rightarrow_{\mathcal{N}} p$ *if and only if p is a member of every non-empty link set.*

(iii) $p \Rightarrow_{\mathcal{N}}$ **false** *if and only if p is not a member of any link set.*

Proof.

(i) By and-closure and or-closure, the fact that **true** occurs in every non-empty link set while **false** never occurs, and the fact that the only logical implications we can have on predicates are derived from: $p \wedge q \Longrightarrow p$, $p \wedge q \Longrightarrow q$, $p \Longrightarrow p \vee q$, $q \Longrightarrow p \vee q$, $p \Longrightarrow p$, $p \Longrightarrow$ **true**, **false** $\Longrightarrow p$, transitivity, $p \Longrightarrow q \wedge r$ if $p \Longrightarrow q$ and $p \Longrightarrow r$, and $p \vee q \Longrightarrow r$ if $p \Longrightarrow r$ and $q \Longrightarrow r$ (using structural induction on derivations).

(ii) By the fact that **true** occurs in every non-empty link set.

(iii) By the fact that **false** never occurs.

□

The following proposition defines the properties of link sets, including the restriction and direct distance operators. The reason for the wording in case (ii) is that we will later extend this proposition to more general sets.

Proposition 3.2 *For every link set S:*

(i) $p \wedge q \in S$ *if and only if* $p \in S$ *and* $q \in S$.

(ii) If $p \in S$ *or* $q \in S$ *then* $p \vee q \in S$.

(iii) $\{\} \downarrow p = \{\}$.

(iv) $S \downarrow \textbf{true} = S$.

(v) $S \downarrow \textbf{false} = \{\}$.

(vi) $S \downarrow p \subseteq S \downarrow q$ *if* $p \Rightarrow_{\mathcal{N}} q$.

(vii) $S \downarrow p = S \downarrow q$ *if* $p \Leftrightarrow_{\mathcal{N}} q$.

(viii) $S \downarrow p \downarrow q = S \downarrow p$ *if* $p \Rightarrow_{\mathcal{N}} q$.

(ix) $S \downarrow p \downarrow p = S \downarrow p$.

(x) $S \downarrow p \downarrow q = S \downarrow p \wedge q$.

(xi) $S \downarrow p \downarrow q = S \downarrow q \downarrow p$.

(xii) $\{\} \,\#\, \delta = \infty$.

(xiii) $(S \downarrow p) \,\#\, \delta \geq S \,\#\, \delta$.

Proof. (i) and (ii) follow by and-closure and or-closure; (iii) by definition 2.3; (iv) and (v) by the fact that **true** occurs in every non-empty link set, but **false** never occurs; (vi) by definitions 2.3 and 2.4; (vii) and (viii) by (vi); (ix) by (viii); (x) by (i) and definition 2.3; (xi) by (x); (xii) and (xiii) by definitions 2.2 and 2.3. □

We extend the definition of the restriction operator to entire networks as follows:

Definition 3.3 $\mathcal{N} \downarrow p$ is the network $\mathcal{N}$ with each link set $\mathcal{L}(A, B)$ replaced by $\mathcal{L}(A, B) \downarrow p$.

Clearly every network $\mathcal{N}$ has this form, since $\mathcal{N} = \mathcal{N} \downarrow \mathbf{true}$. The network $\mathcal{N} \downarrow \mathbf{false}$, on the other hand, has every link set empty. The network $\mathcal{N} \downarrow [\mathbf{from} = A]$ retains only the links from A to other nodes. The network $\mathcal{N} \downarrow \mathbf{supervisor}$ retains only the links $\mathcal{L}(A, B)$ where B is the supervisor of A. The following properties hold:

Proposition 3.4 *For every network $\mathcal{N}$:*

(i) $\mathcal{N} \downarrow p = \mathcal{N} \downarrow q$ *if and only if* $p \Leftrightarrow_{\mathcal{N}} q$.

(ii) $\mathcal{N} \downarrow p \downarrow q = \mathcal{N} \downarrow p$ *if and only if* $p \Rightarrow_{\mathcal{N}} q$.

(iii) $\mathcal{N} \downarrow p \downarrow p = \mathcal{N} \downarrow p$.

(iv) $\mathcal{N} \downarrow p \downarrow q = \mathcal{N} \downarrow p \wedge q$.

(v) $\mathcal{N} \downarrow p \downarrow q = \mathcal{N} \downarrow q \downarrow p$.

Proof.

(i) By proposition 3.2 and definitions 2.3 and 2.4 for the converse of (i).

(ii) By proposition 3.2 and (i), noting that $p \wedge q \Leftrightarrow_{\mathcal{N}} p$.

(iii) and others by proposition 3.2.

□

4 Composition of Link Sets

Given two link sets S and S', we define their composition, written $S; S'$ (also written $S' \circ S$ by most authors) to include predicates which occur in both link sets and distances which are the sum of distances in the two link sets. We use the term *extended link set* for sets which are link sets or which are created by one or more compositions of link sets.

Definition 4.1 The composition of two extended link sets S and S' is defined by:

$$S; S' = \{p \mid p \in S \quad \text{and} \quad p \in S'\} \quad \cup$$
$$\{[\delta : (S \# \delta + S' \# \delta)] \mid \delta \in S \quad \text{and} \quad \delta \in S'\}$$

where $d + \infty = \infty + d = \infty$.

Proposition 4.2 *Composition is associative, i.e.* $S; (S'; S'') = (S; S'); S''$.

Proof. Since addition and logical conjunction are. □

Proposition 4.3 *For every extended link set S, proposition 3.2 still applies.*

Proof. By cases and induction on the number of compositions. □

Note that converse of case (ii) does not apply: it is possible to have $p \vee q \in S; S'$ but not $p \in S; S'$ or $q \in S; S'$ (e.g. when $p \in S$, $q \notin S$, $p \notin S'$, and $q \in S'$).

Proposition 4.4 *For all extended link sets S and S',*

(i) $S; \{\} = \{\} = \{\}; S$.

(ii) $(S; S') \downarrow p = (S \downarrow p); (S' \downarrow p)$.

(iii) $(S; S') \# \delta = S \# \delta + S' \# \delta$.

Proof. Straightforward. □

We further extend link sets by introducing a special set **id** defined as follows:

Definition 4.5 The distinguished extended link set **id** satisfies:

(i) $p \in \mathbf{id}$ for every predicate p.

(ii) $\delta \in \mathbf{id}$ for every distance operator δ.

(iii) $\mathbf{id} \# \delta = 0$ for every distance operator δ.

Proposition 4.6 *The special link set* **id** *satisfies:*

(i) $\mathbf{id} \downarrow p = \mathbf{id}$ *for every predicate p.*

(ii) $\mathbf{id}; S = S = S; \mathbf{id}$ *for every extended link set S.*

(iii) Proposition 3.2 holds for **id** *except for case (v), since* $\mathbf{id} \downarrow \mathbf{false} = \mathbf{id}$

(iv) Proposition 4.4 holds for **id**.

Proof.

(i) By definition 2.3.

(ii) By definition 4.1.

(iii) Trivial.

(iv) Since $\mathbf{id}; \{\} = \{\} = \{\}; \mathbf{id}$, also $(S; \mathbf{id}) \downarrow p = S \downarrow p = (S \downarrow p); (\mathbf{id} \downarrow p)$, and $(S; \mathbf{id}) \# \delta = S \# \delta = S \# \delta + 0 = S \# \delta + \mathbf{id} \# \delta$, and similarly for $\mathbf{id}; S$. □

5 Paths and Categories

Definition 5.1 Given a sequence of nodes $A_1, \ldots, A_n$ we define the *path set* $\mathcal{P}(A) = \mathbf{id}$ for $n = 1$ and as follows for $n \geq 2$:

$$\mathcal{P}(A_1, \ldots, A_n) = \mathcal{S}(A_1, A_2); \mathcal{S}(A_2, A_3); \ldots; \mathcal{S}(A_{n-1}, A_n)$$

where:

$$\begin{aligned} \mathcal{S}(A, B) &= \mathcal{L}(A, B) \quad \text{if } A \neq B \\ &= \mathbf{id} \quad \text{if } A = B \end{aligned}$$

The quantity $\mathcal{P}(A_1, \ldots, A_n) \# \delta$ is the (directed) *distance along the path* $A_1, \ldots, A_n$ under the distance operator δ. We write $\vec{\delta}(A, B)$ for the (directed) distance along paths from A to B, defined to be the minimum over all paths $A, A_2, \ldots, A_{n-1}, B$ of $\mathcal{P}(A, A_2, \ldots, A_{n-1}, B) \# \delta$.

Definition 5.2 Given two (possibly overlapping) sets of nodes $\mathcal{A}$ and $\mathcal{B}$ and a binary relation ρ on nodes, we write $\vec{\delta}_\rho(\mathcal{A}, \mathcal{B})$ for the average, over all pairs $A \in \mathcal{A}$ and $B \in \mathcal{B}$ satisfying $\rho(A, B)$, of $\vec{\delta}(A, B)$.

In the FINC methodology for analysing military organisational structures [15], various performance measures of the form $\vec{\delta}_\rho(\mathcal{A}, \mathcal{B})$ are used. For example, if $\mathcal{A}$ consists of nodes generating information, $\mathcal{B}$ consists of nodes carrying out activities, and $\rho(A, B)$ means that the node A generates information relevant to node B, then $\vec{\delta}_\rho(\mathcal{A}, \mathcal{B})$ represents what the FINC methodology calls the information flow coefficient, which provides a measure of how effectively a military organisation can mobilise information to carry out a task.

Proposition 5.3 *A network $\mathcal{N}$ forms a category, where the nodes $A, B, C, \ldots$ are the objects, and each path set $\mathcal{P}(A_1, \ldots, A_n)$ for $n \geq 1$ is an arrow from A_1 to A_n.*

Proof.

(i) Composition is associative (proposition 4.2).

(ii) **id** acts as an identity for each object A (proposition 4.6).

□

Proposition 5.4 *The relation $\downarrow p$ is a functor mapping the category $\mathcal{N}$ to the category $\mathcal{N} \downarrow p$ for every predicate p.*

Proof.

(i) $\mathbf{id} \downarrow p = \mathbf{id}$ (proposition 4.6).

(ii) $(S; S') \downarrow p = (S \downarrow p); (S' \downarrow p)$ (propositions 4.4 and 4.6).

□

Proposition 5.5 *Let $\mathcal{R}$ be the one-object category (monoid) with the non-negative real numbers and ∞ as arrows, addition as composition, and 0 as identity. Then the relation $\# \delta$ is a functor mapping the category $\mathcal{N}$ to the category $\mathcal{R}$ for every distance operator δ.*

Proof.

(i) $\mathbf{id} \mathbin{\#} \delta = 0$ (definition 4.5).

(ii) $(S; S') \mathbin{\#} \delta = S \mathbin{\#} \delta + S' \mathbin{\#} \delta$ (propositions 4.4 and 4.6).

□

Proposition 5.6 *The collection of functors $\downarrow p$ as arrows and networks $\mathcal{N} \downarrow q$ as objects forms a category (which we denote by $\mathcal{N}^*$) which is isomorphic to the poset category induced by reversing the relation $\Rightarrow_{\mathcal{N}}$.*

Proof. That it is a category is a standard result. For isomorphism, note that the functor $\downarrow q$ from $\mathcal{N} \downarrow p$ to $\mathcal{N} \downarrow p \wedge q$ corresponds to $p \wedge q \Rightarrow_{\mathcal{N}} p$, while in general $p \Rightarrow_{\mathcal{N}} q$ corresponds to the functor $\downarrow p$ from $\mathcal{N} \downarrow q$ to $\mathcal{N} \downarrow p \wedge q = \mathcal{N} \downarrow p$ by proposition 3.4. □

Why are these results significant? There are four main reasons:

1. The fact that our model of a social network forms a well-known mathematical structure (that of a category) acts as a kind of sanity check that our model is reasonable, although our development of this is still preliminary.
2. In future work, more powerful proof techniques from category theory will be used. The use of *fibrations* [1] is one possibility in this regard.
3. Category theory has a close link to programming language semantics [12] and in future work this will allow us to incorporate formal modelling of network updates such as `colour := supervisor ? "red" : "grey"`.
4. Category theory also has close links to logic [10] and this allows us to incorporate modelling of beliefs about networks. Such modelling is critical in our intended application area of analysing international political structures. We give a preliminary version of belief analysis in section 7.

6 Distances

Definition 6.1 For each distance operator δ, we define the (undirected) *distance function* $\delta(A, B)$ on pairs of nodes A and B as the minimum over all paths $A, A_2, \ldots, A_{n-1}, B$ of:

$$\delta^1(A, A_2) + \delta^1(A_2, A_3) + \ldots + \delta^1(A_{n-2}, A_{n-1}) + \delta^1(A_{n-1}, B)$$

where:

$$\begin{aligned} \delta^1(A, B) &= 0 \quad \text{if } A = B \\ &= \min(\mathcal{L}(A, B) \mathbin{\#} \delta,\ \mathcal{L}(B, A) \mathbin{\#} \delta) \quad \text{if } A \neq B \end{aligned}$$

Proposition 6.2 *For every distance operator δ and nodes A, B, and C:*

(i) $0 \leq \delta(A, B) \leq \infty$.

(ii) $\delta(A, B) = \delta(B, A)$.

(iii) $\delta(A, C) \leq \delta(A, B) + \delta(B, C)$.

Proof. By definition. For (iii), not all paths go via B. □

Note that in general distance functions are not metrics, since we may have $\delta(A, B) = 0$ for $A \neq B$ when $\delta = 0$ in some link sets.

Definition 6.3 We write $\delta \downarrow p$ to mean the distance function δ in $\mathcal{N} \downarrow p$, i.e. such that:

$$S \# (\delta \downarrow p) = (S \downarrow p) \# \delta$$

Hence a given δ can be restricted to follow only links satisfying p. We are interested in comparing different concepts of distance (δ *vs* δ') and also the same concept of distance based on different subsets of links ($\delta \downarrow p$ *vs* $\delta \downarrow q$).

Figure 1 shows a symmetric social network, based on the the ground force structure during the Gulf War [3,8]. Boxes represent division-level units from participating countries (in the case of Saudi, Kuwaiti, and Gulf state units, these are notional), while circles represent commanders. Units on the right were under the control of Norman Schwarzkopf, and those on the left under the control of Saudi Prince Khaled bin Sultan. There are two forms of distance: δ_{cul} and δ_{com}. Links between division-level units (not shown in figure 1) contain the distance operator δ_{cul} measuring cultural differences. These range from $\frac{1}{8}$ for units from the same country and service to 6 for the less than friendly relationship between the US and Syria. Cultural differences between the US Army and Marines are reflected by a distance of $\frac{1}{2}$. Dark grey lines in the figure show formal command relationships, and these correspond to link sets containing the basic predicate **formal** and $\delta_{\text{com}} = 1$. The light grey line between the US VII Corps commander and the Egyptian Corps commander represents an informal working relationship. This corresponds to a link set containing the basic predicate **informal** and the slightly greater distance $\delta_{\text{com}} = 2$. Inspection of the diagram shows that when δ_{com} is extended to a distance function between division-level units, it ranges from 2 to 6.

Physical distance in figure 1 indicates a combination of the two distances, as produced by a spring-embedding layout algorithm. When all pairs of division-level units are considered, there is a statistical correlation of 0.6 between the distance functions δ_{cul} and δ_{com}. This indicates that the organisational structure negotiated between the US and Saudi Arabia was fairly successful in separating culturally different units.

This is more clearly illustrated in figure 2, where each division-level unit is represented by a pair of boxes (one white, one coloured) linked by an arrow. We call this a *social flow diagram*. As a result of the spring-embedding layout algorithm, the physical distance between white boxes closely indicates δ_{cul} (physical distance has a 0.97 correlation with δ_{cul}), while the physical distance between coloured boxes indicates δ_{com} (somewhat less closely, with a correlation of 0.86). The arrows indicate how culturally similar units have been separated in some cases, and culturally dissimilar units have been combined in others.

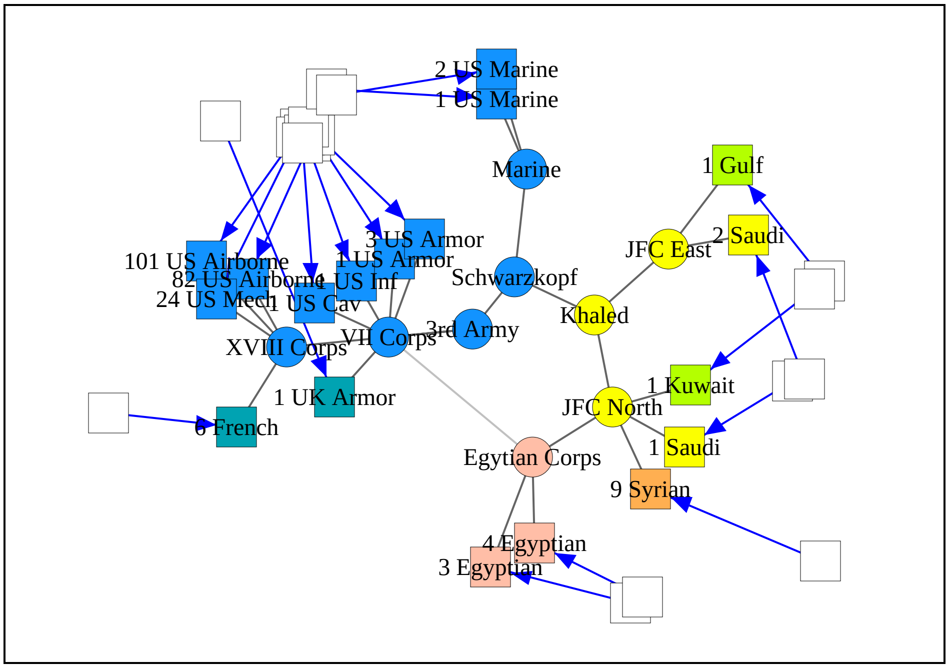

Fig. 2. Social Flow Diagram for Gulf War Ground Forces

Proposition 6.4 *Let $D_{\mathcal{N}}$ be the poset category whose objects are all distance functions δ or $\delta\downarrow p$ on $\mathcal{N}$ and whose arrows from δ_1 to δ_2 denote that $\delta_1(A,B) \leq \delta_2(A,B)$ for all A and B. Let F_δ map $\mathcal{N}^*$ to $D_{\mathcal{N}}$ by mapping $\mathcal{N}\downarrow p$ to the distance function $\delta\downarrow p$ and the functor $\downarrow q$ from $\mathcal{N}\downarrow p$ to $\mathcal{N}\downarrow p\wedge q$ to the unique arrow from $\delta\downarrow p$ to $\delta\downarrow p\wedge q$. Then F_δ is well-defined and is a functor.*

Proof. By definition 6.3 and proposition 3.2, $(\delta\downarrow p)(A,B) \leq (\delta\downarrow p\wedge q)(A,B)$. The rest is straightforward. □

In future work we plan to study the interaction between predicates and distance by examining properties of the functors F_δ.

7 Truth and Belief

We now extend the definition of $p \Rightarrow_{\mathcal{N}} q$ above (definition 2.4) to more general *statements* $\theta, \phi, \psi, \ldots$ of the form p, $\theta\wedge\phi$, $\theta\vee\phi$, $\theta\Rightarrow\phi$, or $\delta(A,B)=d$:

Definition 7.1 We define $\mathcal{N}\models\theta$ (θ is true in $\mathcal{N}$) as follows:

(i) $\mathcal{N}\models\delta(A,B)=d$ if and only if $\delta(A,B)=d$ in $\mathcal{N}$.

(ii) $\mathcal{N}\models\theta\wedge\phi$ if and only if $\mathcal{N}\models\theta$ and $\mathcal{N}\models\phi$.

(iii) $\mathcal{N}\models\theta$ if and only if $\mathcal{L}(A,B)\models_{\mathcal{N}}\theta$ for every $\mathcal{L}(A,B)\neq\{\}$ in $\mathcal{N}$, otherwise.

(iv) $S\models_{\mathcal{N}} p$ if and only if $p\in S$.

(v) $S\models_{\mathcal{N}}\theta\wedge\phi$ if and only if $S\models_{\mathcal{N}}\theta$ and $S\models_{\mathcal{N}}\phi$.

(vi) $S \models_{\mathcal{N}} \theta \vee \phi$ if and only if $S \models_{\mathcal{N}} \theta$ or $S \models_{\mathcal{N}} \phi$.

(vii) $S \models_{\mathcal{N}} \theta \Rightarrow \phi$ if and only if $S \models_{\mathcal{N}} \theta$ implies $S \models_{\mathcal{N}} \phi$.

(viii) $S \models_{\mathcal{N}} \delta(A, B) = d$ if and only if $\delta(A, B) = d$ in $\mathcal{N}$.

Proposition 7.2 *For all predicates p and q,*

(i) $\mathcal{N} \models p$ if and only if **true** $\Rightarrow_{\mathcal{N}} p$*.*

(ii) $\mathcal{N} \models p \Rightarrow q$ if and only if $p \Rightarrow_{\mathcal{N}} q$.

Proof. By definition 2.4, proposition 3.1, and and-closure/or-closure of link sets. □

In other words, we really have extended the definition of $p \Rightarrow_{\mathcal{N}} q$. We incorporate the notion of *belief* about networks by using Kripke (possible worlds) semantics [2,5,6,11,13]. Each person or entity P is associated with a predicate p such that for a network $\mathcal{N}$, P believes the network is actually $\mathcal{N} \downarrow p$, i.e. the functor $\downarrow p$ acts as an *accessibility* relation between possible worlds which is transitive (by proposition 3.2, therefore corresponding to doxastic or K4 belief logic), but not reflexive (since beliefs may be incorrect):

Definition 7.3 We define $\mathcal{N} \models P$ **believes** θ (P believes θ about $\mathcal{N}$) by:

$$\mathcal{N} \models P \textbf{ believes } \theta \quad \text{if and only if} \quad \mathcal{N} \downarrow p \models \theta$$

where the person or entity P is associated with the predicate p.

For two important categories of belief, we can find an alternative characterisation of what it means for a person or entity to believe something:

Proposition 7.4 *For all statements θ,*

(i) If there is no occurrence of $\delta(A, B) = d$ in θ, then $\mathcal{N} \models P$ **believes** *θ if and only if $\mathcal{N} \models p \Rightarrow \theta$.*

(ii) $\mathcal{N} \models P$ **believes** *$\delta(A, B) = d$ if and only if $(\delta \downarrow p)(A, B) = d$ in $\mathcal{N}$.*

where the person or entity P is associated with the predicate p.

Proof.

(i) By definitions 7.1 and 7.3

(ii) By definitions 6.3, 7.1 and 7.3

□

Let $\mathcal{N}_G$ represent the network in figure 1, let E represent either of the Egyptian Divisions and let U represent any of the US divisions in VII Corps. Because of the informal link (with $\delta_{\text{com}} = 2$) between the VII Corps commander (General Fred Franks) and the Egyptian Corps commander (Major General Salah Halabi), we have $\mathcal{N}_G \models \delta_{\text{com}}(E, U) = 4$. However, Norman Schwarzkopf seemed to only be aware of the formal links, i.e. links in $\mathcal{N}_G \downarrow$ **formal**. Therefore, since $\mathcal{N}_G \downarrow$ **formal** $\models \delta_{\text{com}}(E, U) = 7$, we conclude that Schwarzkopf believed the command distance δ_{com} to be greater than it really was: $\mathcal{N}_G \models$ Schwarzkopf **believes** $\delta_{\mathrm{com}}(E, U) = 7$. As a result of this, Schwarzkopf exaggerated liaison problems between VII Corps and the Egyptian units on its right flank [3], and so held the US 1$^{\mathrm{st}}$ Cavalry in reserve longer than was necessary.

Thus we see that our category-theoretic formalisation allows us to reason formally about beliefs relating to social and organisational networks. In future work we will introduce more complex models of belief, using the same category-theoretic framework. We intend to implement logical analysis of belief as we have outlined it here within our CAVALIER tool, in the same way that in previous work we have automated belief logic for cryptographic protocol analysis [5]. Our goal in doing this is to provide practical assistance to military commanders assembling coalition forces, and also to analyse coalition forces assembled by other countries.

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